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RECURSION SEQUENCES

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. И. Маркушевич

ВОЗВРАТНЫЕ ПОСЛЕДОВАТЕЛЬНОСТИ

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V. Zhitomirsky, D.Sc. (Tech.)

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PREFACE

The subject "Recursion Sequences" is related with the secondary school curriculum (arithmetic and geometric progressions, the sequence of squares of natural numbers, the sequence of coefficients in the quotient of two polynomials in ascending powers etc.). It is at the same time a real though a small mathematical theory*, complete and clear like everything created by the eminent masters of analysis who founded it.

The fundamentals of the theory of recursion sequences were worked out and published in the twenties of the XVIII-th century by the French mathematician Abraham De Moivre [the formula $(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$ bears his name] and by one of the first members of the Petersburg Academy of Sciences, the Swiss mathematician Daniel Bernulli. A comprehensive theory of these sequences is due to the great XVIII-th century mathematician, member of the Petersburg Academy academician Leonard Euler who devoted to the recursion sequences (series) Chapter 13 of his *Introduction to the Analysis of Infinitesimals* (1748). Among the recent works the courses of the calculus of finite differences read by the renowned Russian mathematicians academicians P.L. Chebyshev and A. A. Markov should be especially noted; the courses include notes on the theory of recursion sequences.

* For the reader who is versed in analysis it can be stated that the theory is a strict analogue of the theory of linear differential equations with constant coefficients.

1. The concept of a recursion sequence is a broad generalization of the concept of an arithmetic or geometric progression. Its particular cases also include the sequences of squares or cubes of natural numbers, the sequences of digits of a rational repeating decimal (and any recurrent sequence in general), the sequences of coefficients in a quotient of two polynomials in ascending powers of χ etc. It follows that in the course of mathematical studies at a secondary school recursion sequences are very often met with. The theory of recursion sequences is a special part of the so called *calculus of finite differences*. This theory will be expounded here for readers with no previous experience in the subject (in only one place we shall refer without proof to a general proposition established by the theory of linear algebraic equations).

2. Sequences will be written in the form

$$u_1,\ u_2,\ u_3,\ \dots,\ u_n,\ \dots, \tag{1}$$

or, shortly, $\{u_n\}$. If there is a natural number k, and numbers $a_1, a_2, \dots, a_k$ (real or imaginary), such that starting from a certain number n and for all following numbers

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\dots+a_ku_n \quad (n\geqslant m\geqslant 1), \tag{2}$$

the sequence (1) is called a *recursion sequence of order k* and the relation (2), a *recursion relation of order k*.

Thus, a recursion sequence is characterized by each of its terms (starting from a certain one) being expressed by the same number k of terms immediately preceding it, according to formula (2). The very name "recursion sequence" is used just because one must turn back to the preceding terms in order to obtain the following one. Some examples of recursion sequences will be given below.

Example 1. *Geometric Progression*. For the geometric progression

$$u_1=a,\ u_2=aq,\ u_3=aq^2,\ \dots,\ u_n=aq^{n-1},\ \dots; \tag{3}$$

relation (2) takes the form

$$u_{n+1}=qu_n \tag{4}$$

where $k=1$ and $a_1=q$. Thus the geometric progression is a recursion sequence of the *first* order.

Example 2. *Arithmetic Progression.* For the arithmetic progression

$$u_1 = a, \ u_2 = a + d, \ u_3 = a + 2d, \ \ldots, \ u_n = a + (n-1)d, \ \ldots$$

we have

$$u_{n+1} = u_n + d.$$

This relation is not of the form of relation (2). Characteristic of the latter is that its right-hand member contains only sequence terms with constant coefficients. However, considering two relations for two consecutive values of n:

$$u_{n+2} = u_{n+1} + d \quad \text{and} \quad u_{n+1} = u_n + d$$

and subtracting the latter from the former term by term we obtain

$$u_{n+2} - u_{n+1} = u_{n+1} - u_n,$$

or

$$u_{n+2} = 2u_{n+1} - u_n. \tag{5}$$

Relation (5) is of the form of (2). This suggests that the arithmetic progression is a recursion sequence of the *second* order.

Example 3. Consider the ancient problem of Fibonacci * about the number of rabbits. Required is the number of pairs of adult rabbits descending from one pair in the course of a year if it is known that each adult pair of rabbits gives birth to a new pair every month, the newborn becoming fully adult in a month. The interest of the problem lies not in its solution, which is easily arrived at, but in the sequence whose terms express the total number of adult pairs of rabbits at the initial moment (u_1), in a month (u_2), in two months (u_3) and in general after n months (u_{n+1}). Obviously $u_1 = 1$. One newborn pair will be added a month later, but the number of adult pairs will be the same: $u_2 = 1$. In two months (from the initial moment) the little rabbits will become adult and the total number of adult pairs will equal 2: $u_3 = 2$. Assume that we have calculated the number of adult pairs after

* Fibonacci, or Leonardo da Pisa, a medieval Italian mathematician (circa 1170), wrote the book *Liber Abaci*, containing extensive arithmetic and geometric information borrowed from peoples of Middle Asia and Byzantium and creatively reworked and developed by him.

$n-1$ months, u_n, and after n months, u_{n+1}. By this time the u_n adult pairs that existed before will have an offspring of u_n additional pairs, so that after $n+1$ months the total number of adult pairs will be

$$u_{n+2} = u_{n+1} + u_n. \tag{6}$$

Hence

$$u_4 = u_3 + u_2 = 3, \quad u_5 = u_4 + u_3 = 5,$$
$$u_6 = u_5 + u_4 = 8, \quad u_7 = u_6 + u_5 = 13, \ \ldots$$

Thus the sequence obtained is

$$\left.\begin{array}{l} u_1 = 1, \ u_2 = 1, \ u_3 = 2, \ u_4 = 3, \\ u_5 = 5, \ u_6 = 8, \ u_7 = 13, \ \ldots, \end{array}\right\} \tag{7}$$

i.e. every following term is equal to the sum of two preceding terms. This sequence is known as the *Fibonacci sequence* (series) and its terms, as *Fibonacci numbers*. Relation (6) shows that the Fibonacci sequence is a recursion sequence of the *second* order.

Example 4. Consider now the sequence of squares of natural numbers:

$$u_1 = 1^2, \quad u_2 = 2^2, \quad u_3 = 3^2, \ \ldots, \ u_n = n^2, \ \ldots \tag{8}$$

where $u_{n+1} = (n+1)^2 = n^2 + 2n + 1$. Thus

$$u_{n+1} = u_n + 2n + 1. \tag{9}$$

Increasing n by unity we have

$$u_{n+2} = u_{n+1} + 2n + 3. \tag{10}$$

And hence [subtracting (9) from (10) term by term]

$$u_{n+2} - u_{n+1} = u_{n+1} - u_n + 2,$$

or

$$u_{n+2} = 2u_{n+1} - u_n + 2. \tag{11}$$

Increasing in equality (11) n by unity gives

$$u_{n+3} = 2u_{n+2} - u_{n+1} + 2. \tag{12}$$

Hence [subtracting (11) from (12) term by term]

$$u_{n+3} - u_{n+2} = 2u_{n+2} - 3u_{n+1} + u_n,$$

or

$$u_{n+3} = 3u_{n+2} - 3u_{n+1} + u_n. \tag{13}$$

The recursion relation obtained is of the *third* order. Hence sequence (8) is a recursion sequence of the *third* order. Similarly it can be verified that the sequence of the cubes of natural numbers

$$1^3,\ 2^3,\ 3^3,\ \ldots,\ n^3,\ \ldots \tag{14}$$

is a recursion sequence of the *fourth* order. Its terms satisfy the equation

$$u_{n+4}=4u_{n+3}-6u_{n+2}+4u_{n+1}-u_n, \tag{15}$$

and the reader is invited to derive it.

Example 5. All recurrent sequences are recursion sequences. Consider for instance the sequence of digits of the rational repeating decimal

$$\frac{761}{1332}=0{,}57132132132\ \cdots$$

I . this sequence

$$\left.\begin{matrix} u_1=5,\ u_2=7,\ u_3=1,\ u_4=3, \\ u_5=2,\ u_6=1,\ u_7=3,\ \ldots \end{matrix}\right\} \tag{16}$$

It is obvious that

$$u_{n+3}=u_n \quad (n\geqslant 3). \tag{17}$$

Rewriting equation (17) to give it the form of relation (2) we have

$$u_{n+3}=0\cdot u_{n+2}+0\cdot u_{n+1}+1\cdot u_n.$$

Thus, Eq. (17) is a recursion relation of the *third* order ($k=3$, $a_1=0$, $a_2=0$, $a_3=1$), and sequence (16) is a recursion sequence of the *third* order.

Example 6. Now let us consider the sequence whose terms are the coefficients of the quotient of two polynomials arranged in ascending powers of x. Given

$$P(x)=A_0+A_1x+\ldots+A_lx^l$$

and

$$Q(x)=B_0+B_1x+\ldots+B_kx^k \quad (B_0\neq 0).$$

Let us divide $P(x)$ by $Q(x)$; if there is a remainder the division can be infinitely continued. The quotient will contain terms in the following order:

$$D_0+D_1x+D_2x^2+D_3x^3+\ldots+D_nx^n+\ldots$$

Consider the sequence

$$u_1=D_0,\ u_2=D_1,\ \ldots,\ u_n=D_{n-1},\ \ldots \tag{18}$$

We shall prove that this is a sequence of order k (remember that k is the power of the divisor). In order to do it we choose an arbitrary natural number n satisfying only one condition $n\geqslant l-k+1$, and in dividing we stop at the term of the quotient with x^{n+k}. The remainder will be a polynomial $R(x)$ containing powers of x higher than $n+k$. Writing down the relation between dividend, divisor, quotient and remainder we obtain the following identity:

$$A_0+\ldots+A_lx^l=$$
$$=(B_0+\ldots+B_kx^k)\cdot(D_0+\ldots+D_{n+k}x^{n+k})+R(x).$$

Find the coefficients of x^{n+k} in the right-hand and left-hand members of the identity and equate them. As $n+k\geqslant l+1$, the coefficient of x^{n+k} in the left-hand member is zero. Hence the coefficient of x^{n+k} in the right-hand member is also zero. But the terms with x^{n+k} are present only in the product $(B_0+\ldots+B_kx^k)\cdot(D_0+\ldots+D_{n+k}x^{n+k})$. (The remainder $R(x)$, as was mentioned above, contains only higher powers of x). Thus the required coefficient is

$$D_{n+k}B_0+D_{n+k-1}B_1+\ldots+D_nB_k. \tag{19}$$

According to the foregoing argument this coefficient is equal to zero:

$$D_{n+k}B_0+D_{n+k-1}B_1+\ldots+D_nB_k=0.$$

Hence (as $B_0\neq 0$)

$$D_{n+k}=-\frac{B_1}{B_0}D_{n+k-1}-\ldots-\frac{B_k}{B_0}D_n \quad (n\geqslant l-k+1). \tag{20}$$

This is a recursion relation of order k. It follows that sequence (18) is a recursion sequence of order k.

3. Of all the above examples Example 6 is of the most general character. We shall now show that any *arbitrary recursion sequence of the k-th order*

$$u_1,\ u_2,\ \ldots,\ u_n,\ \ldots, \tag{21}$$

satisfying the equation of the form

$$u_{n+k}=a_1u_{n+k-1}+\ldots+a_ku_n \quad (n\geqslant m\geqslant 1), \tag{22}$$

coincides with the sequence whose terms are the coefficients of the quotient of a polynomial $P(x)$ by the polynomial

$$Q(x)=1-a_1x-\dots-a_kx^k. \quad (23)$$

Let n be an arbitrary natural number satisfying the condition $n>k+m-2$; multiplying the polynomial $Q(x)$ by $u_1+u_2x+u_3x^2+\dots+u_{n+1}x^n$ we obtain

$$\begin{aligned}&(1-a_1x-a_2x^2-\dots-a_kx^k)(u_1+u_2x+\\&+\dots+u_{k+m-1}x^{k+m-2}+\dots+u_{n+1}x^n)=\\&=[u_1+(u_2-a_1u_1)x+\dots+\\&+\dots+(u_{k+m-1}-a_1u_{k+m-2}-\dots-a_ku_{m-1})x^{k+m-2}]+\\&+[(u_{k+m}-a_1u_{k+m-1}-\dots-a_ku_m)x^{k+m-1})+\\&+\dots+(u_{n+1}-a_1u_n-\dots a_ku_{n-k+1})x^n]-\\&-[(a_1u_{n+1}+\dots+a_ku_{n-k+2})x^{n+1}+\dots+a_ku_{n+1}x^{n+k}]. \quad (24)\end{aligned}$$

The polynomial in the first pair of square brackets is of a degree not higher than $l=k+m-2$; the coefficients in it are independent of the chosen number n. Let us call it $P(x)$:

$$\begin{aligned}P(x)=u_1+(u_2-a_1u_1)x+\\+\dots+(u_{k+m-1}-a_1u_{k+m-2}-\dots-a_ku_{m-1})x^{k+m-2}. \quad (25)\end{aligned}$$

All the coefficients in the polynomial in the next pair of brackets are zero by Eq. (22). Lastly, the third pair of brackets embraces a polynomial whose coefficients depend on n; it contains no terms of a power lower than $n+1$.

Denoting this polynomial by $R_n(x)$ we can rewrite identity (24) in the following form:

$$\begin{aligned}P(x)=(1-a_1x-a_2x^2-\dots-a_kx^k)(u_1+u_2x+\\+\dots+u_{n+1}x^n)+R_n(x). \quad (26)\end{aligned}$$

Hence, it is obvious that $u_1+u_2x+\dots+u_{n+1}x^n$ is the quotient and $R_n(x)$, the remainder of the division of $P(x)$ by

$$Q(x)=1-a_1x-a_2x^2-\dots-a_kx^k$$

i. e.

$$u_1,\ u_2,\ \dots,\ u_n,\ u_{n+1},\ \dots$$

is indeed the sequence of coefficients of the quotient of polynomial (25) by (23).

Consider, for example, the Fibonacci sequence

$$u_1=1,\ u_2=1,\ u_3=2,\ u_4=3,\ u_5=5,\ \dots$$

Since its terms satisfy the relation

$$u_{n+2}=u_{n+1}+u_n \quad (n\geqslant 1),$$

then $m=1$, $k=2$, $a_1=1$, $a_2=1$ and $Q(x)=1-x-x^2$.

The highest power in $P(x)$ must not exceed $k+m-2=1$. By formula (25) we have

$$P(x)=1+(1-1\times 1)x=1.$$

Thus, the Fibonacci numbers coincide with the terms of the sequence of coefficients of the quotient of 1 by $1-x-x^2$.

4. One of the problems which are offered for solution to the students of secondary schools when studying the arithmetic and geometric progressions and the sequence of squares of natural numbers is to find the sum of n terms of each of these sequences.

Let

$$u_1,\ u_2,\ \dots,\ u_n,\ \dots \quad (27)$$

be a recursion sequence of order k whose terms satisfy the relation

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\dots+a_ku_n \quad (n\geqslant m). \quad (28)$$

Consider a new sequence formed by the sums s_n of numbers (27);

$$s_1=u_1,\ s_2=u_1+u_2,\ \dots,\ s_n=u_1+u_2+\dots+u_n,\ \dots \quad (29)$$

We shall show that the sequence of these sums is also a recursion sequence of order $k+1$ and its terms satisfy the relation

$$\begin{aligned}s_{n+k+1}=(1+a_1)s_{n+k}+\\+(a_2-a_1)s_{n+k-1}+\dots+(a_k-a_{k-1})s_{n+1}-a_ks_n. \quad (30)\end{aligned}$$

Note that

$$\left.\begin{aligned}u_1=s_1,\quad u_1,=s_2-u_1=s_2-s_1,\ \dots\\ \dots,\ u_n=s_n-(u_1+\dots+u_{n-1})=s_n-s_{n-1},\ \dots\end{aligned}\right\} \quad (31)$$

Assuming $s_0=0$ so that $u_1=s_1-s_0$ and substituting in equation (28) for $u_1, u_2, \ldots, u_n$ their expressions in terms of $s_0, s_1, \ldots, s_n$ we obtain

$$s_{n+k}-s_{n+k-1}=a_1(s_{n+k-1}-s_{n+k-2})+ \\ +a_2(s_{n+k-2}-s_{n+k-3})+\ldots+a_k(s_n-s_{n-1}),$$

whence

$$s_{n+k}=(1+a_1)s_{n+k-1}+(a_2-a_1)s_{n+k-2}+\ldots+ \\ +(a_k-a_{k-1})s_n-a_ks_{n-1} \quad (n\geqslant m),$$

or replacing n by $n+1$ we obtain

$$s_{n+k+1}=(1+a_1)s_{n+k}+(a_2-a_1)s_{n+k-1}+\ldots+ \\ +(a_k-a_{k-1})s_{n+1}-a_ks_n \quad (n\geqslant m-1).$$

This is a recursion relation of order $k+1$. Several examples are given below.

(a) *Geometric Progression*. In a geometric progression $u_n=aq^{n-1}$ and $s_n=u_1+u_2+\ldots+u_n=a+aq+\ldots aq^{n-1}$. Since the terms of $\{u_n\}$ satisfy the relation $u_{n+1}=qu_n$, the terms of $\{s_n\}$ must satisfy the relation

$$s_{n+2}=(1+q)s_{n+1}-qs_n. \tag{32}$$

(b) *The Sequence of Squares of Natural Numbers*. Here $u_n=n^2$ and $s_n=1+2^2+\ldots+n^2$. Since the terms of $\{u_n\}$ satisfy the relation

$$u_{n+3}=3u_{n+2}-3u_{n+1}+u_n$$

(cf. p. 9)

$$s_{n+4}=4s_{n+3}-6s_{n+2}+4s_{n+1}-s_n.$$

(c) *Fibonacci Numbers*. Since the Fibonacci numbers satisfy the relation

$$u_{n+2}=u_{n+1}+u_n$$

their sums s_n must satisfy the relation

$$s_{n+3}=2s_{n+2}-s_n.$$

5. In the case of the simplest recursion sequences, for instance, arithmetic and geometric progressions, sequences of cubes or squares of natural numbers, and recursion sequences, any term can be found without calculating the preceding terms. In the case of the Fibonacci number sequence or of the general sequence of coefficients in the quotient of two polynomials it seems, at first sight, to be impossible to do that. In order to calculate, say, the thirteenth Fibonacci number we preliminarily find all the preceding terms (using the relation $u_{n+2}=u_{n+1}+u_n$):

$$u_1=1,\ u_2=1,\ u_3=2,\ u_4=3,\ u_5=5,\ u_6=8, \\ u_7=13,\ u_8=21,\ u_9=34,\ u_{10}=55,\ u_{11}=89, \\ u_{12}=144,\ u_{13}=233.$$

We now turn to a detailed study of the structure of recursion sequence terms. This study results in formulas which make it possible to calculate in the most general case any terms of a recursion sequence without resorting to the calculation of the preceding terms. These formulas can be regarded as far reaching generalizations of the formulas expressing the general term of an arithmetic or geometric progression.

Consider the following recursion relation of order k:

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\ldots+a_ku_n. \tag{33}$$

If it holds for any value of natural numbers $n=1, 2, 3, \ldots$, then assuming $n=1$ we can obtain

$$u_{k+1}=a_1u_k+a_2u_{k-1}+\ldots+a_ku_1.$$

Hence, knowing $u_1, u_2, \ldots, u_k$, we can now find u_{k+1}. Assuming $n=2$ in relation (33) we obtain

$$u_{k+2}=a_1u_{k+1}+a_2u_k+\ldots+a_ku_2.$$

So now we know the value of u_{k+2} too. In general if m is a natural number and we have calculated the terms

$$u_1,\ u_2,\ \ldots,\ u_k,\ u_{k+1},\ \ldots,\ u_{m+k-1}$$

of a sequence, then assuming $n=m$ in relation (33) we find the next term u_{m+k}.

Thus the terms of a recursion sequence of order k which satisfy relation (33) are unambiguously determined by this equation if the first k terms of the sequence $u_1, u_2, \ldots, u_k$ are known. Choosing the sequences in various ways (there are no conditions limiting this choice) we can obtain an infinite number of different sequences which satisfy rela-

tion (33). The distinction between them will manifest itself in the first k terms (at least in one of them) and will be observed in the following terms too.

For example, the relation of the first order

$$u_{n+1} = qu_n$$

is satisfied by any of the geometric progressions whose common ratio is q (they differ in the value of the first term u_1); the relation of the second order

$$u_{n+2} = 2u_{n+1} - u_n$$

(or $u_{n+2} - u_{n+1} = u_{n+1} - u_n$) is satisfied by any of the arithmetic progressions which differ in at least one of the terms $u_1 = a$ and $u_2 = a + d$; i. e. differ either in the value of the first term (a) or in the value of the common difference (d) or in both.

Now consider another relation of the second order

$$u_{n+2} = u_{n+1} + u_n$$

Besides the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

where $u_1 = u_2 = 1$ this equation is satisfied by many other sequences obtained by sampling various values of u_1 and u_2. For example, with $u_1 = -3$ and $u_2 = 1$ we obtain the following sequence:

$$-3, 1, -2, -1, -3, -4, -7, -11, -18, -29, \ldots$$

Let us have a certain number of sequences which satisfy the same relation (33)

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_n, \ldots, \\ y_1, y_2, \ldots, y_n, \ldots, \\ \cdots\cdots\cdots\cdots \\ z_1, z_2, \ldots, z_n, \ldots, \end{array}\right\} \quad (34)$$

The following equations will hold:

$$\left.\begin{array}{l} x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \ldots + a_k x_n, \\ y_{n+k} = a_1 y_{n+k-1} + a_2 y_{n+k-2} + \ldots + a_k y_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_{n+k} = a_1 z_{n+k-1} + a_2 z_{n+k-2} + \ldots + a_k z_n. \end{array}\right\} \quad (35)$$

Taking as many arbitrary numbers $A, B, \ldots, C$, as sequences (34), we multiply each term of the first of equations (35) by A, of the second, by B, ..., of the last, by C and add the products:

$$\begin{aligned} Ax_{n+k} + By_{n+k} + \ldots + Cz_{n+k} = \\ = a_1 (Ax_{n+k-1} + By_{n+k-1} + \ldots + Cz_{n+k-1}) + \\ + a_2 (Ax_{n+k-2} + By_{n+k-2} + \ldots + Cz_{n+k-2}) + \\ + \ldots + a_k (Ax_n + By_n + \ldots + Cz_n). \end{aligned} \quad (36)$$

It follows that the sequence

$$\left.\begin{array}{l} t_1 = Ax_1 + By_1 + \ldots + Cz_1, \\ t_2 = Ax_2 + By_2 + \ldots + Cz_2, \\ \cdots\cdots\cdots\cdots \\ t_n = Ax_n + By_n + \ldots + Cz_n, \end{array}\right\} \quad (37)$$

obtained from sequences (34) by multiplying each term of the first sequence by A, of the second by B, ..., of the last by C and then adding the products termwise (first terms with the first ones, second terms with the second ones etc.) satisfies relation (33). Since the choice of the numbers $A, B \ldots, C$ is arbitrary we can by changing them obtain, generally speaking, different values of the terms $t_1, t_2, t_3, \ldots$.

Now let the sequence

$$u_1, u_2, \ldots, u_n, \ldots \quad (38)$$

be such that it satisfies relation (33); the question is: can we give the numbers $A, B, \ldots, C$ the values such that the first k terms of sequence (37) coincide with the first k terms of sequence (38)? If this can be achieved then according to the preceding argument all the terms of sequences (37) and (38) will also coincide, i. e. we will have for any natural number n:

$$u_n = Ax_n + By_n + \ldots + Cz_n \quad (39)$$

Thus it becomes possible (as yet hypothetically) to express *any* of the infinite number of sequences which satisfy the same recursion relation of order k in terms of some of sequences (34), by formula (39). The realization of this possibility depends on whether we can select the numbers

$A, B, \ldots, C$ so that the equations

$$\left.\begin{array}{l} Ax_1 + By_1 + \ldots + Cz_1 = u_1, \\ Ax_2 + By_2 + \ldots + Cz_2 = u_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Ax_k + By_k + \ldots + Cz_k = u_k \end{array}\right\} \qquad (40)$$

with arbitrary given values of their right-hand members $u_1, u_2, \ldots, u_k$ be satisfied.

Since the numbers $A, B, \ldots, C$ are in this case the unknowns and the number of equations is equal to the k-th order of the recursion relation, it follows, that the number of the unknowns $A, B, \ldots, C$ [which coincides with the number of sequences (34)] should also be taken equal to k. It is known that a system of k algebraic equations (40) in k unknowns $A, B, \ldots, C$ can have solutions depending on the kind of the coefficients in the system: $x_1, y_1, \ldots, z_1, \ldots, x_k, y_k, \ldots, z_k$, i. e. on the kind of the first terms of sequences (34). The solutions will certainly exist, with arbitrary right-hand members $u_1, u_2, \ldots, u_k$ if we assume, for instance, that

$$\left.\begin{array}{l} x_1 = 1, \; y_1 = 0, \; \ldots, \; z_1 = 0; \\ x_2 = 0, \; y_2 = 1, \; \ldots, \; z_2 = 0; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_k = 0, \; y_k = 0, \; \ldots, \; z_k = 1 \end{array}\right\} \qquad (41)$$

Indeed, in this case system (40) takes a very simple form, and the solution is directly obvious:

$$\left.\begin{array}{l} A = u_1, \\ B = u_2, \\ \cdot \quad \cdot \quad \cdot \\ C = u_k \end{array}\right\}$$

Of course a different choice of numbers is possible with which system (40) has a solution whatever the right-hand members of the equations. Assume, for instance,

$$\left.\begin{array}{l} x_1 = 1, \; y_1 = 1, \; \ldots, \; z_1 = 1; \\ x_2 = 0, \; y_2 = 1, \; \ldots, \; z_2 = 1; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_k = 0, \; y_k = 0, \; \ldots, \; z_k = 1. \end{array}\right\} \qquad (42)$$

The system now takes the form

$$\left.\begin{array}{r} A + B + \ldots + C = u_1, \\ B + \ldots + C = u_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ C = u_k. \end{array}\right\}$$

Hence we obtain

$$C = u_k, \quad \ldots, \quad B = u_2 - u_3, \quad A = u_1 - u_2.$$

We now turn to the general case and state the following theorem:

For the system of k linear algebraic equations (40) *in k unknowns to have one and only one solution $A, B, \ldots, C$ with any values of the right-hand members $u_1, u_2, \ldots, u_k$ it is necessary and sufficient that the corresponding homogeneous system*

$$\left.\begin{array}{l} Ax_1 + By_1 + \ldots + Cz_1 = 0, \\ Ax_2 + By_2 + \ldots + Cz_2 = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Ax_k + By_k + \ldots + Cz_k = 0 \end{array}\right\} \qquad (40')$$

have one only zero solution: *

$$A = B = \ldots = C = 0$$

The reader will easily ascertain that the condition in the formulation of the theorem is fulfilled in the particular

* This is a convenient proposition since its application does not require any knowledge of the theory of determinants. The reader who is familiar with the theory will remember that for system (40) to have a solution with any values of the right-hand members of these equations it is necessary and sufficient that the determinant of the system

$$\Delta = \begin{vmatrix} x_1 & y_1 & \ldots & z_1 \\ x_2 & y_2 & \ldots & z_2 \\ \cdot & \cdot & \cdot & \cdot \\ x_n & y_n & \ldots & z_n \end{vmatrix}$$

be different from zero. The same condition is necessary and sufficient for system (40) to have one and only one solution for any given right-hand members (say, equal to zero). Thus for a system of k linear equations in k unknowns the condition for the existence of a solution with any right-hand members coincides with the condition that one and only one solution with zero right-hand members exists. Just this fact is expressed by the statement in the text.

cases (41) and (42). Later on we shall meet with cases in which the proposition stated above will prove useful. Until then we shall simply make use of the fact (established irrespective of the theorem) that there always exist numbers $x_1, \ldots, z_1, \ldots, x_k, \ldots, z_k$ [the first terms of sequence (34)] such that the system of equations (40) has a solution with any values of $u_1, u_2, \ldots, u_k$.

If such numbers have been chosen as the first terms of sequences (34), then according to the foregoing any sequence which satisfies the recursion relation (33) can be expressed by formula (39) in which the numbers $A, B, \ldots, C$ are found from equations (40). The system of k sequences (34) by means of which the terms of any sequence satisfying relation (33) can be expressed by formulas (39) (i. e. by multiplying the terms by certain numbers $A, B, \ldots, C$, and adding the products) is called the *basis* of the recursion sequence.

It follows from the above sequence that every recursion sequence has a basis which can be selected in different ways. For example, systems whose first terms are

$$\underbrace{\left.\begin{matrix} 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & 0 \\ . & . & . & . \\ 0, & 0, & & 1 \end{matrix}\right\}}_{(k)}, \quad \text{or} \quad \underbrace{\left.\begin{matrix} 1, & 1, & \ldots, & 1 \\ 0, & 1, & \ldots, & 1 \\ . & . & . & . \\ 0, & 0, & \ldots, & 1 \end{matrix}\right\}}_{(k)}$$

form the basis of an arbitrary recursion sequence of the k-th order.

Let us sum up the results of **5**.

For every recursion relation of order k there exists an infinite number of different sequences which satisfy this relation. Any such sequence can be formed from k sequences which satisfy the relation and form its basis by multiplying each of the k-th sequences by certain corresponding numbers A, B, ..., C respectively and adding the products termwise.

Therefore in order to obtain a full solution of a recursion relation of order k it is sufficient to find but a finite number k of the sequences that satisfy it and form its basis.

We now give some examples in order to make this formulation clear.

Example 1. Let we have the following recursion relation of the second order:

$$u_{n+2} = 2u_{n+1} - u_n.$$

Its basis must consist of two sequences:

$$x_1, x_2, x_3, \ldots, x_n, \ldots,$$
$$y_1, y_2, y_3, \ldots, y_n, \ldots$$

In selecting them we shall assume

$$x_1 = 1, \; x_2 = \quad \text{and} \quad y_1 = 0, \; y_2 = 1.$$

Since rewriting the recursion relation in the form

$$u_{n+2} - u_{n+1} = u_{n+1} - u_n,$$

we see that the difference of two successive terms of the sequence is constant, i. e. the sequence which satisfies this relation, is necessarily an arithmetic progression, then the sequence $\{x_n\}$ whose first terms are $x_1 = 1$ and $x_2 = 1$ is an arithmetic progression with zero common difference, i. e.

$$1, 1, 1, \ldots, 1, \ldots \quad (x_n = 1)$$

and the sequence $\{y_n\}$ with $y_1 = 0$ and $y_2 = 1$ is an arithmetic progression with the common difference equal to unity, i. e.

$$0, 1, 2, \ldots, n-1 \ldots \quad (y_n = n-1).$$

A term of any recursion sequence which satisfies this equation can be expressed in the form

$$u_n = Ax_n + By_n = A + B(n-1)$$

where A and B can be determined from the equations

$$u_1 = A + B(1-1),$$
$$u_2 = A + B(2-1),$$

or

$$u_1 = A,$$
$$u_2 = A + B.$$

Hence

$$A = u_1, \quad B = u_2 - u_1$$

it follows that

$$u_n = u_1 + (n-1)(u_2 - u_1).$$

This is the general formula for a term of any recursion sequence satisfying the relation

$$u_{n+2}=2u_{n+1}-u_n.$$

Denoting $u_1=a$, $u_2-u_1=d$ we can give it the following form:

$$u_n=a+(n-1)d.$$

This is the well known formula for the general term of the arithmetic progression.

Example 2. Consider another recursion relation of the second order:

$$u_{n+2}=u_{n+1}+u_n.$$

Assuming $x_1=1$, $x_2=1$ we obtain the familiar Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, \ldots$$

As a second sequence entering into the basis let us take the sequence $\{y_n\}$ in which $y_1=0$ and $y_2=1$. We can write then

$$y_3=y_2+y_1=1, \quad y_4=y_3+y_2=2, \quad y_5=y_4+y_3=3, \ldots$$

In this sequence $y_2=x_1$, $y_3=x_2$, $y_4=x_3$, $y_5=x_4 \ldots$ and in general $y_n=x_{n-1}$ $(n=2, 3, \ldots)$. Indeed, since we have found that these equalities hold for any value of $n\leqslant m+1$ so that in particular $y_{m+1}=x_m$, $y_m=x_{m-1}$, then for y_{m+2} we shall have

$$y_{m+2}=y_{m+1}+y_m=x_m+x_{m-1}=x_{m+1}$$

and the equalities also hold for $n=m+2$.

Thus

$$y_n=x_{n-1} \quad (n=2, 3, \ldots).$$

Therefore for any sequence, which satisfies the relation

$$u_{n+2}=u_{n+1}+u_n,$$

we find [formula (39)]:

$$u_n=Ax_n+By_n,$$

where A and B are found from the equations:

$$u_1=Ax_1+By_1=A,$$
$$u_2=Ax_2+By_2=A+B.$$

Hence

$$A=u_1, \quad B=u_2-u_1$$

and

$$u_n=u_1x_n+(u_2-u_1)y_n.$$

With $n\geqslant 2$ x_{n-1} can be substituted for y_n; then

$$u_n=u_1x_n+(u_2-u_1)x_{n-1} \quad (n\geqslant 2);$$

or

$$u_n=u_1(x_n-x_{n-1})+u_2x_{n-1}.$$

With $n\geqslant 3$

$$x_n=x_{n-1}+x_{n-2}, \quad \text{or} \quad x_n-x_{n-1}=x_{n-2},$$

and consequently

$$u_n=u_1x_{n-2}+u_2x_{n-1} \quad (n\geqslant 3).$$

Thus terms of any sequence $\{u_n\}$, which satisfy the relation

$$u_{n+2}=u_{n+1}+u_n,$$

can be expressed in terms of Fibonacci numbers by the formula we have established. In particular if $u_1=-3$, $u_2=1$ (cf. p. 16) then

$$u_n=-3x_{n-2}+x_{n-1} \quad (n\geqslant 3).$$

6. It will be shown now that under some very general conditions the basis of the recursion relation (33)

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\ldots+a_ku_n,$$

comprising k geometric progressions with different common ratios can be found. In order to do it let us determine under what conditions the geometric progression

$$x_1=1, \quad x_2=q, \quad \ldots, x_n=q^{n-1}, \quad \ldots \quad (q\neq 0)$$

satisfies ralation (33). Noting that

$$x_{n+k}=q^{n+k-1}, \quad x_{n+k-1}=q^{n+k-2}, \ldots, x_n=q^{n-1},$$

we substitute these quantities in equation (33) (for u_{n+k}, u_{n+k-1}, $\ldots$, u_n) and obtain:

$$q^{n+k-1}=a_1q^{n+k-2}+a_2q^{n+k-3}+\ldots+a_nq^{n-1},$$

whence

$$q^k = a_1 q^{k-1} + a_2 q^{k-2} + \ldots + a_k. \tag{43}$$

Thus a *geometric progression can satisfy the recursion relation (33) of order k if and only if the common ratio q of the progression satisfies the algebraic equation (43) of degree k with the same coefficients as those in equation (33).*

Equation (43) is called the *characteristic* equation for the recursion relation (33). If $q=\alpha$ is a root of the characteristic equation (real or imaginary) then assuming

$$x_n = \alpha^{n-1} \quad (n = 1, 2, \ldots), \tag{44}$$

we obtain a geometric progression whose first term is $x_1 = 1$ and with a common ratio α; this progression satisfies relation (33). Since α is a root of equation (43) we have, indeed,

$$\alpha^k = a_1\alpha^{k-1} + a_2\alpha^{k-2} + \ldots + a_k.$$

Multiplying the two members of this expression by α^{n-1} where n is an arbitrary natural number we obtain which

$$\alpha^{n+k-1} = a_1\alpha^{n+k-2} + a_2\alpha^{n+k-3} + \ldots + a_k\alpha^{n-1},$$

means that sequence (44) satisfies relation (33).

Thus to each root $q=\alpha$ of the characteristic equation (43) there is a corresponding geometric progression (44) with a common ratio α; this progression satisfies the recursion relation (33).

To form the basis of geometric progressions with different common ratios only one must have a sufficient number k of them and consequently have k *different* roots of the characteristic equation.

Assume that all the roots of the characteristic equation are different:

$$q_1 = \alpha, \quad q_2 = \beta, \ \ldots, \ q_k = \gamma,$$

We have then k geometric progressions which satisfy relation (33):

$$\left.\begin{array}{l} 1, \ \alpha, \ \alpha^2, \ \ldots, \ \alpha^{n-1}, \ \ldots, \\ 1, \ \beta, \ \beta^2, \ \ldots, \ \beta^{n-1}, \ \ldots, \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ 1, \ \gamma, \ \gamma^2, \ \ldots, \ \gamma^{n-1}, \ \ldots \end{array}\right\} \tag{45}$$

We shall show now that the system of sequences (45) is the basis of relation (33), i. e. that for any sequence $\{u_n\}$ which satisfies relation (33) we can find numbers $A, B, \ldots, C$ such that for any n we shall have

$$u_n = A\alpha^{n-1} + B\beta^{n-1} + \ldots + C\gamma^{n-1}. \tag{46}$$

In order to prove this it suffices to verify that the system of equations obtained from (45) with $n = 1, 2, \ldots, k$

$$\left.\begin{array}{l} A + B + \ldots + C = u_1, \\ A\alpha + B\beta + \ldots + C\gamma = u_2, \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ A\alpha^{k-1} + B\beta^{k-1} + \ldots + C\gamma^{k-1} = u_k, \end{array}\right\} \tag{47}$$

has a solution with respect to the unknowns $A, B, \ldots, C$ with any values of the right-hand members of these equations; for this it is sufficient, in turn, (cf. p. 19) that the corresponding homogeneous system

$$\left.\begin{array}{l} A + B + \ldots + C = 0, \\ A\alpha + B\beta + \ldots + C\gamma = 0, \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ A\alpha^{k-1} + B\beta^{k-1} + \ldots + C\gamma^{k-1} = 0 \end{array}\right\} \tag{48}$$

have one and only one zero solution. That is actually so.

In fact, suppose there exists a non-zero solution of (48), i. e. there are numbers $A, B, \ldots, C$ such that at least one of them, say, A is different from zero, and the numbers satisfy system (48). To reduce this proposition to a contradiction let us first form a polynomial $M(x)$ of degree $k-1$ which vanishes with $x = \beta, \ldots, x = \gamma$ and equals unity with $x = \alpha$. Since this polynomial is of degree $k-1$ and vanishes with $k-1$ different values of x: $\beta, \ldots, \gamma$, it must be of the following form:

$$M(x) = \mu(x-\beta) \ldots (x-\gamma),$$

where μ is some number. Taking $x=\alpha$, we must obtain $M(\alpha) = 1$; hence

$$1 = \mu(\alpha-\beta) \ldots (\alpha-\gamma)$$

and

$$\mu = \frac{1}{(\alpha-\beta) \ldots (\alpha-\gamma)}.$$

Thus

$$M(x)=\frac{(x-\beta)\ldots(x-\gamma)}{(\alpha-\beta)\ldots(\alpha-\gamma)};$$

and obviously the polynomial satisfies the given conditions. Removing the parentheses and collecting like terms we can give the polynomial the following form:

$$M(x)=m_0+m_1x+\ldots+m_{k-1}x^{k-1}.$$

If we multiply now equations (48) by $m_0, m_1, \ldots, m_{k-1}$ and add them term by term we obtain

$$\begin{aligned}&A(m_0+m_1\alpha+\ldots+m_{k-1}\alpha^{k-1})+\\&+B(m_0+m_1\beta+\ldots+m_{k-1}\beta^{k-1})+\ldots+\\&+C(m_0+m_1\gamma+\ldots+m_{k-1}\gamma^{k-1})=0,\end{aligned}$$

or

$$AM(\alpha)+BM(\beta)+\ldots+CM(\gamma)=0.$$

Since $M(\alpha)=1$, $M(\beta)=0, \ldots, M(\gamma)=0$ it follows that

$$A=0,$$

which contradicts our assumption.

Thus system (48) has one and only one zero solution and system (47) must have a solution (one only) with any $u_1, u_2, \ldots, u_k$, and this, in turn, means, that system (45) is the basis of recursion relation (33).

Thus we have found that for every recursion relation

$$u_{n+k}=a_1u_{n+k-1}+\ldots+a_ku_n,$$

whose corresponding characteristic equation

$$q^k=a_1q^{k-1}+a_2q^{k-2}+\ldots+a_k$$

has different roots: $q=\alpha$, $q=\beta$, $\ldots$, $q=\gamma$, there exists a basis consisting of k geometric progressions with the common ratios: $\alpha, \beta, \ldots, \gamma$. In other words for terms of any sequence $\{u_n\}$ which satisfies relation (33) there are k numbers: $A, B, \ldots, C$ [they are found from equations (47)] such that

$$u_n=A\alpha^{n-1}+B\beta^{n-1}+\ldots+C\gamma^{n-1}\quad(n=1, 2, 3, \ldots).$$

Let us sum up the ideas of **6.**

A recursion relation of order k has a corresponding algebraic equation of degree k with the same coefficients—its characteristic equation. Each root of the characteristic equation is the common ratio of a geometric progression which satisfies the recursion relation. If all roots of the characteristic equation are different, k geometric progressions are obtained which constitute the basis of the recursion relation. It follows that in this case terms of any sequence which satisfies the recursion relation can be obtained by adding term by term certain geometric progressions (k in number).

7. We now turn to the application of the results obtained. Consider first the Fibonacci sequence. Its recursion relation is as follows:

$$u_{n+2}=u_{n+1}+u_n$$

and the characteristic equation (43) accordingly takes the form

$$q^2=q+1.$$

Solving this equation we obtain two different real roots:

$$\alpha=\frac{1}{2}+\frac{1}{2}\sqrt{5}\quad\text{and}\quad\beta=\frac{1}{2}-\frac{1}{2}\sqrt{5}.$$

The general term of the Fibonacci sequence can therefore be written in the following form:

$$u_n=A\alpha^{n-1}+B\beta^{n-1}.$$

In order to find the unknown coefficients A and B assume $n=1$ and $n=2$; we obtain

$$\left.\begin{aligned}&u_1=1=A+B\\&u_2=1=A\alpha+B\beta=\frac{1}{2}(A+B)+\frac{\sqrt{5}}{2}(A-B).\end{aligned}\right\}$$

Solving this system we find:

$$A=\frac{\sqrt{5}+1}{2\sqrt{5}},\quad B=\frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Hence

$$u_n=\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}+\frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}.$$

or

$$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{49}$$

This is precisely the general expression for the Fibonacci numbers. At first sight the formula obtained looks unwieldy and poorly suited for calculations. However using it one can obtain many interesting results. We shall show, for instance, that the sum of squares of two adjacent Fibonacci numbers is a Fibonacci number as well.

Indeed

$$u_n^2=\frac{1}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n}+\left(\frac{1-\sqrt{5}}{2}\right)^{2n}-2(-1)^n\right]$$

$$u_{n+1}^2=\frac{1}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+2}+\left(\frac{1-\sqrt{5}}{2}\right)^{2n+2}-2(-1)^{n+1}\right];$$

Hence

$$u_{n+1}^2+u_n^2=$$
$$=\frac{1}{5}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2n}\cdot\frac{5+\sqrt{5}}{2}+\left(\frac{1-\sqrt{5}}{2}\right)^{2n}\cdot\frac{5-\sqrt{5}}{2}\right\}=$$
$$=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right\}=u_{2n+1}.$$

It follows that

$$u_{n+1}^2+u_n^2=u_{2n+1} \tag{50}$$

For example

$$u_{13}=u_7^2+u_6^2=13^2+8^2=233.$$

By the way, this is the answer to the Fibonacci problem.

We leave it for the reader to prove a more general relation for the Fibonacci numbers than (50), viz.

$$u_nu_m+u_{n+1}u_{m+1}=u_{n+m+1}. \tag{51}$$

The following theorem will be proved as another application of formula (49):

For any two natural numbers a and b with $a<b$, the number of consecutive processes in the Euclidean algorithm which are necessary to find the greatest common divisor (G.C.D.) of a and b is not higher than five times the number of digits in a written down using the decimal system.

Applying the Euclidean algorithm to find the G.C.D. of the numbers b and a we obtain a chain of equalities:

$$\left.\begin{array}{ll}(1) & b=ax'+y',\\ (2) & a=y'x''+y'',\\ (3) & y'=y''x'''+y''',\\ & \dots\\ (k) & y^{(k-2)}=y^{(k-1)}x^{(k)}+y^{(k)},\\ (k+1) & y^{(k-1)}=y^{(k)}x^{(k+1)}\end{array}\right\} \tag{52}$$

The consecutive remainders satisfy the inequalities

$$a>y'>y''>y'''>\ldots>y^{(k-1)}>y^{(k)}\geqslant 1.$$

The remainder in the last of equalities (52) is zero.

This means that the foregoing remainder $y^{(k)}$ is the required G.C.D. of the numbers b and a. Therefore k is the number of operations required for finding the G.C.D. Our problem, as we have said, consists in evaluating the number k. For this purpose let us compare the numbers $y^{(k)}$, $y^{(k-1)}$, ..., y', a with the Fibonacci numbers $u_1, u_2, u_3, \ldots$. Note that $y^{(k)}\geqslant 1=u_2$ but the foregoing remainder $y^{(k-1)}$ is greater than $y^{(k)}$, and hence $y^{(k-1)}\geqslant 2=u_3$. Therefore we conclude from the equality k that

$$y^{(k-2)}=y^{(k-1)}x^{(k)}+y^{(k)}\geqslant y^{(k-1)}\cdot 1+y^{(k)}\geqslant u_3+u_2=u_4$$

Thus

$$y^{(k)}\geqslant u_2,\quad y^{(k-1)}\geqslant u_3,\quad y^{(k-2)}\geqslant u_4.$$

Assume that we have proved the following inequalities to be valid

$$y^{(k)}\geqslant u_2,\ \ldots,\ y^{(m)}\geqslant u_{k-m+2},\ y^{(m-1)}\geqslant u_{k-m+3}$$
$$(m-1\geqslant 2).$$

Then from the equality $y^{(m-2)}=y^{(m-1)}x^m+y^m$ we obtain

$$y^{(m-2)}\geqslant y^{(m-1)}\cdot 1+y^{(m)}\geqslant u_{k-m+3}+u_{k-m+2}=u_{k-m+4}.$$

Continuing this argument we come to the inequalities

$$y''\geqslant u_k,\quad y'\geqslant u_{k+1}$$

and further from (52, 2) we deduce that

$$a=y'x''+y''\geqslant y'\cdot 1+y''\geqslant u_{k+1}+u_k=u_{k+2}.$$

By formula (49) the term u_{k+2} takes the form

$$u_{k+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right].$$

Therefore

$$a\geqslant\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]>$$

$$>\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-1\right] \qquad (53)$$

$\left(\text{since } \left|\frac{1-\sqrt{5}}{2}\right|<1, \text{ and consequently } \left|\frac{1-\sqrt{5}}{2}\right|^{k+2}<1\right).$

From (53) we find that

$$\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}<a\sqrt{5}+1<\sqrt{5}(a+1)<\left(\frac{1+\sqrt{5}}{2}\right)^{2}(a+1)$$

$$\left(\sqrt{5}<\left(\frac{1+\sqrt{5}}{2}\right)^{2}=\frac{3+\sqrt{5}}{2}, \text{ as } \sqrt{5}<3\right).$$

Therefore

$$\left(\frac{1+\sqrt{5}}{2}\right)^{k}<a+1. \qquad (54)$$

We now note that

$$u_5=5=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{5}-\left(\frac{1-\sqrt{5}}{2}\right)^{5}\right]<$$

$$<\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{5}+1\right].$$

Hence

$$\left(\frac{1+\sqrt{5}}{2}\right)^{5}>5\sqrt{5}-1>10.$$

Consequently

$$10^{k}<\left(\frac{1+\sqrt{5}}{2}\right)^{5k}<(a+1)^{5}. \qquad (55)$$

If number a is written down in the decimal system with n digits (a is an n-digit number) it is obvious that

$$10^{n-1}\leqslant a<10^{n};$$

hence

$$a+1\leqslant 10^{n}$$

and thus by virtue of inequality (55)

$$10^{k}<(a+1)^{5}\leqslant 10^{5n}$$

or

$$k<5n. \qquad (56)$$

This is precisely the required result: the number of consecutive k divisions in the Euclidean algorithm is less than five times the number of digits in the smaller of the two numbers, b, and a, written in the decimal system. The foregoing proof shows that the most unfavourable case of the application of the Euclidean algorithm (as to the number of operations near the limit established by the theorem) is that in which b and a are the adjacent Fibonacci numbers. In order to confirm this statement let us take, for instance, $b=u_{20}=6765$ and $a=u_{19}=4181$. Here a is a four-digit number and by the above theorem the number of operations in the Euclidean algorithm must be less than $5\times4=20$. In fact we have to perform in this case $k=17$ operations:

(1) $6765=4181\times1+2584$	(10) $89=55\times1+34$
(2) $4181=2584\times1+1597$	(11) $55=34\times1+21$
(3) $2584=1597\times1+987$	(12) $34=21\times1+13$
(4) $1597=987\times1+610$	(13) $21=13\times1+8$
(5) $987=610\times1+377$	(14) $13=8\times1+5$
(6) $610=377\times1+233$	(15) $8=5\times1+3$
(7) $377=233\times1+144$	(16) $5=3\times1+2$
(8) $233=144\times1+89$	(17) $3=2\times1+1$
(9) $144=89\times1+55$	(18) $2=1\times2+0$

Fibonacci numbers in descending order are obtained here one after another as the remainders of division. All quotients (except the last) are unity and this is why the number of operations is so great. The greatest common divisor proved to be unity [equality (17)] which could be foreseen beforehand for the adjacent Fibonacci numbers. Indeed, from $u_{n+2}=u_{n+1}+u_n$ it follows that the G.C.D. of the numbers u_{n+2} and u_{n+1} is the same as the G.C.D. of u_{n+1} and u_n. Therefore for any pair of adjacent Fibonacci numbers the G.C.D. is the same. In order to find it one can just consider the pair $u_2=u_1=1$ which shows that the G.C.D. is unity.

8. Let us consider as the next example the recurrent sequence (16):

$$u_1=5,\ u_2=7,\ u_3=1,\ u_4=3,\ u_5=2,\ u_6=1,\ u_7=3,\ \ldots$$

The recursion relation in this case is

$$u_{n+3}=u_n \quad (n \geqslant 3),$$

and thus the characteristic equation is

$$q^3=1.$$

The roots of this equation are

$$\alpha=1,\ \beta=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ and } \gamma=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

Therefore the general term of the sequence should be assumed to have the following form [cf. formula (46)]:

$$u_n=A\alpha^{n-1}+B\beta^{n-1}+C\gamma^{n-1}=$$
$$=A+B\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{n-1}+C\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{n-1}$$

We can require this formula to be valid for all values of n with which our recursion relation holds too: $n=3$, 4, 5, ...

Note that

$$-\frac{1}{2}+i\frac{\sqrt{3}}{2}=-\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right),$$
$$-\frac{1}{2}-i\frac{\sqrt{3}}{2}=-\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right).$$

Therefore by De Moivre's theorem (formula)

$$u_n=A+B\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{n-1}+C\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{n-1}=$$
$$=A+(-1)^{n-1}B\left[\cos\frac{\pi}{3}(n-1)-i\sin\frac{\pi}{3}(n-1)\right]+$$
$$+(-1)^{n-1}C\left[\cos\frac{\pi}{3}(n-1)+i\sin\frac{\pi}{3}(n-1)\right]=$$
$$=A+(B+C)(-1)^{n-1}\cos\frac{\pi}{3}(n-1)+$$
$$+i(-B+C)(-1)^{n-1}\sin\frac{\pi}{3}(n-1).$$

Let us have $B+C=A_1$ and $i(-B+C)=A_2$; then the above formula will be rewritten in the form

$$u_n=A+A_1(-1)^{n-1}\cos\frac{\pi}{3}(n-1)+A_2(-1)^{n-1}\sin\frac{\pi}{3}(n-1)$$
$$(n\geqslant 3),$$

and we have now to find only the unknown coefficients A, A_1 and A_2. Taking $n=3$, $n=4$ and $n=5$ we obtain three equations in three unknowns:

$$u_3=1=A+A_1\cos\frac{2\pi}{3}+A_2\sin\frac{2\pi}{3}=A-\frac{1}{2}A_1+\frac{\sqrt{3}}{2}A_2,$$
$$u_4=3=A-A_1\cos\frac{3\pi}{3}-A_2\sin\frac{3\pi}{3}=A+A_1,$$
$$u_5=2=A+A_1\cos\frac{4\pi}{3}+A_2\sin\frac{4\pi}{3}=A-\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2.$$

Hence we find

$$A=2,\quad A_1=1 \quad\text{and}\quad A_2=-\frac{1}{\sqrt{3}}.$$

Thus

$$u_n=2+(-1)^{n-1}\left[\cos(n-1)\frac{\pi}{3}-\frac{1}{\sqrt{3}}\sin(n-1)\frac{\pi}{3}\right]=$$
$$=2+(-1)^n\cdot\frac{2}{\sqrt{3}}\sin(n-2)\frac{\pi}{3}\quad(n\geqslant 3).$$

It can be seen that in this example the general term of the sequence is expressed by trigonometric functions which is in agreement with the recurrent character of the sequence.

As the last example consider the one directly related to the division of polynomials.

Given two polynomials: $P(x)=3+x^2-x^5$ and $Q(x)=-2-x-2x^2+x^3$. The problem is to find the structure of the coefficients in the quotient of $P(x)$ by $Q(x)$. The sequence of the coefficients in the quotient

$$u_1=D_0,\ u_2=D_1,\ \ldots,\ u_n=D_{n-1},\ \ldots$$

as noted in 2, is a recursion sequence whose terms satisfy equation (20):

$$D_{n+k}=-\frac{B_1}{B_0}D_{n+k-1}\ldots-\frac{B_k}{B_0}D_n\quad(n\geqslant l-k+1),$$

where k is the degree of $Q(x)$, $B_0, B_1, \ldots, B_k$ are the coefficients of $Q(x)$ and l is the degree of $P(x)$.

This means that $k=3, B_0=2, B_1=-1, B_2=-2, B_3=1, l=5$:

$$D_{n+3}=\frac{1}{2}D_{n+2}+\frac{2}{2}D_{n+1}-\frac{1}{2}D_n \quad (n\geqslant 5-3+1=3),$$

i. e.

$$D_{n+3}=\frac{1}{2}D_{n+2}+D_{n+1}-\frac{1}{2}D_n \quad (n\geqslant 3).$$

The characteristic equation takes the form

$$q^3=\frac{1}{2}q^2+q-\frac{1}{2},$$

or

$$q^3-q-\frac{1}{2}(q^2-1)=\left(q-\frac{1}{2}\right)(q-1)(q+1)=0.$$

Therefore its roots are

$$\alpha=\frac{1}{2},\quad \beta=1,\quad \gamma=-1,$$

and for D_n we obtain the formula

$$D_n=A\cdot\left(\frac{1}{2}\right)^n+B\cdot 1^n+C(-1)^n \quad (n\geqslant 3).$$

Assuming $n=3$, $n=4$, and $n=5$ we obtain three equations:

$$D_3=\frac{1}{8}A+B-C,$$

$$D_4=\frac{1}{16}A+B+C,$$

$$D_5=\frac{1}{32}A+B-C.$$

In these equations not only the coefficients A, B, and C, but also the numbers D_3, D_4, D_5 are unknown. In order to find the unknowns let us perform the division of $P(x)$ by $Q(x)$ up to the point where the quotient will contain terms of the fifth power, inclusive.

Dividing we get

$$\begin{array}{l|l}
3+x^2-x^5 & 2-x-2x^2+x^3 \\ \hline
-\;3-\frac{3}{2}x-3x^2+\frac{3}{2}x^3 & \frac{3}{2}+\frac{3}{4}x+2\frac{3}{8}x^2+1\frac{3}{16}x^3+2\frac{19}{32}x^4 \\ \cline{1-1}
\quad\frac{3}{2}x+4x^2-\frac{3}{2}x^3-x^5 & \\
-\;\frac{3}{2}x-\frac{3}{4}x^2-\frac{3}{2}x^3+\frac{3}{4}x^4 & \\ \cline{1-1}
\qquad 4\frac{3}{4}x^2-\frac{3}{4}x^4-x^5 & \\
-\;4\frac{3}{4}x^2-2\frac{3}{8}x^3-4\frac{3}{4}x^4+2\frac{3}{8}x^5 & \\ \cline{1-1}
\qquad\quad 2\frac{3}{8}x^3+4x^4-3\frac{3}{8}x^5 & \\
-\;2\frac{3}{8}x^3-1\frac{3}{16}x^4-2\frac{3}{8}x^5+1\frac{3}{16}x^6 & \\ \cline{1-1}
\qquad\qquad 5\frac{3}{16}x^4-x^5-1\frac{3}{16}x^6 & \\
-\;5\frac{3}{16}x^4-2\frac{19}{32}x^5-5\frac{3}{16}x^6+2\frac{19}{32}x^7 & \\ \cline{1-1}
\qquad\qquad\quad 1\frac{19}{32}x^5+4x^6-2\frac{19}{32}x^7. &
\end{array}$$

Hence

$$D_0=\frac{3}{2},\quad D_1=\frac{3}{4},\quad D_2=2\frac{3}{8},\quad D_3=1\frac{3}{16},\quad D_4=2\frac{19}{32},\quad D_5=\frac{51}{64}$$

and the system of equations obtained above takes the form

$$\frac{1}{8}A+B-C=1\frac{3}{16},$$

$$\frac{1}{16}A+B+C=2\frac{19}{32},$$

$$\frac{1}{32}A+B-C=\frac{51}{64},$$

whence we find

$$A=4\frac{1}{6},\quad B=\frac{3}{2},\quad C=\frac{5}{6}.$$

Thus

$$D_n=4\frac{1}{6}\cdot\frac{1}{2^n}+\frac{3}{2}+\frac{5}{6}(-1)^n \quad (n\geqslant 3).$$

The problem is solved. It follows by the above formula

$$D_6=2\frac{51}{128},\quad D_7=\frac{179}{256},\quad D_8=2\frac{179}{512},\quad \ldots$$

9. In all of the above examples the characteristic equation had only simple roots. Now consider the example pertaining to the sequence whose terms are the sums of squares of natural numbers (given on p. 15). The recursion relation for this sequence is

$$s_{n+4}=4s_{n+3}-6s_{n+2}+4s_{n+1}-s_n,$$

and accordingly its characteristic equation takes the form

$$q^4=4q^3-6q^2+4q-1,$$

or

$$q^4-4q^3+6q^2-4q+1=(q-1)^4=0.$$

It has only one quadruple root: $q=1$; therefore we obtain in this case only one geometric progression with the common ratio 1 whose terms satisfy the above recursion relation.

In such cases we have to find other simple recursion sequences which can form, together with this geometric progression, the basis of our equation. Such sequences in our example are:

$$0,\ 1,\ 2,\ 3,\ \ldots,\ n-1,\ \ldots;$$
$$0,\ 1,\ 4,\ 9,\ \ldots,\ (n-1)^2,\ \ldots;$$
$$0,\ 1,\ 8,\ 27,\ \ldots,\ (n-1)^3,\ \ldots.$$

(the reader can easily verify this assertion). Without treating this very general case which would require rather unwieldy calculations we shall consider the following typical example.

Let the recursion relation be

$$u_{n+k}=C_k^{k-1}\alpha u_{n+k-1}-C_k^{k-2}\alpha^2u_{n+k-2}+\ \ldots\ +(-1)^{k-1}C_k^0\alpha^ku_n, \tag{57}$$

where C_k^{k-1}, C_k^{k-2}, $\ldots$, C_k^0 are binomial coefficients of the k-th order. The corresponding characteristic equation

$$q^k=C_k^{k-1}\alpha q^{k-1}-C_k^{k-2}\alpha^2q^{k-2}+\ldots+(-1)^{k-1}C_k^0\alpha^k$$

can be written in the form

$$(q-\alpha)^k=0.$$

It has a k-fold root $q=\alpha$; obviously

$$(\alpha-\alpha)^k=\alpha^k-C_k^{k-1}\alpha^k+C_k^{k-2}\alpha^k-\ldots+(-1)^kC_k^0\alpha^k=0. \tag{58}$$

Consider in general the following identities

$$(\alpha-\alpha)^{k-m}=\alpha^{k-m}-C_{k-m}^{k-m-1}\alpha^{k-m}+$$
$$+C_{k-m}^{k-m-2}\alpha^{k-m}-\ldots+(-1)^{k-m}C_{k-m}^0\alpha^{k-m}=0,$$

where $m=0,\ 1,\ 2,\ \ldots,\ k-1$ or

$$(1-1)^{k-m}=C_{k-m}^{k-m}-C_{k-m}^{k-m-1}+C_{k-m}^{k-m-2}-\ldots+$$
$$+(-1)^{\mu}C_{k-m}^{k-m-\mu}+\ldots+(-1)^{k-m}C_{k-m}^0=0. \tag{59}$$

Equality (59) corresponding to $m=0$ takes the form

$$C_k^k-C_k^{k-1}+C_k^{k-2}-\ldots+(-1)^{\mu}C_k^{k-\mu}+$$
$$+\ldots+(-1)^kC_k^0=0. \tag{59'}$$

Noting that

$$C_k^{k-\mu}=\frac{k(k-1)\ldots(\mu+1)}{1\cdot 2\ldots(k-\mu)}=\frac{k(k-1)\ldots(k-m+1)}{(k-m-\mu+1)\ldots(k-\mu)}C_{k-m}^{k-m-\mu}$$
$$(m=1,\ 2,\ \ldots,\ k-1;\ 0\leqslant\mu\leqslant k-m),$$

or

$$k(k-1)\ldots(k-m+1)C_{k-m}^{k-m-\mu}=$$
$$=(k-m-\mu+1)\ldots(k-\mu)C_k^{k-\mu}, \tag{60}$$

we multiply each of the equalities (59′) by the corresponding multiplier $k(k-1)\ldots(k-m+1)$. We can now using (60) write them in the form

$$(k-m+1)\ldots kC_k^k-(k-m)\ldots(k-1)C_k^{k-1}+\ldots+$$
$$+(-1)^{\mu}(k-m-\mu+1)\ldots(k-\mu)C_k^{k-\mu}+$$
$$+\ldots+(-1)^{k-m}1\cdot 2\ldots mC_k^0=0 \tag{59''}$$
$$(m=1,\ 2,\ \ldots,\ k-1).$$

We shall now prove that the following equations hold with $m=0,\ 1,\ 2,\ \ldots,\ k-1$:

$$k^mC_k^k-(k-1)^mC_k^{k-1}+\ldots+(-1)^{\mu}(k-\mu)^mC_k^{k-\mu}+$$
$$+\ldots+(-1)^k\cdot 0^mC_k^0=0. \tag{61}$$

Indeed, the equality corresponding to $m=0$ coincide with (59′) and thus holds true.

Assume, reasoning by induction, that equalities (61) have already been proved for $m=0,\ 1,\ \ldots,\ j$ $(j\leqslant k-2)$. Let us demonstrate that the equality which corresponds to $m=j+1$ also holds true. In order to prove it we shall

make use of a polynomial of degree $j+1$:

$$f(x)=(x-j)(x-j+1)\ldots(x-1)x=$$
$$=x^{j+1}-\beta_j x^j-\ldots-\beta_1 x. \qquad (62)$$

Multiplying (61) with $m=1, 2, \ldots, j$ by the corresponding numbers $\beta_1, \beta_2, \ldots, \beta_j$ we obtain

$$\left.\begin{aligned}&\beta_1 k C_k^k-\beta_1(k-1)C_k^{k-1}+\\&+\ldots+\beta_1(-1)^\mu(k-\mu)C_k^{k-\mu}+\\&+\ldots+\beta_1(-1)^k\cdot 0\cdot C_k^0=0,\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&\beta_j k^j C_k^k-\beta_j(k-1)^j C_k^{k-1}+\\&+\ldots+\beta_j(-1)^\mu(k-\mu)^j C_k^{k-\mu}+\\&+\ldots+\beta_j(-1)^k\cdot 0\cdot C_k^0=0.\end{aligned}\right\} \qquad (63)$$

Let us rewrite equality (59″) corresponding to the value $m=j+1$ in the form

$$f(k)C_k^k-f(k-1)C_k^{k-1}+\ldots+(-1)^\mu f(k-\mu)C_k^{k-\mu}+$$
$$+\ldots+(-1)^k f(0)C_k^0=0. \qquad (64)$$

In doing that we made use of the equalities

$$(k-j)\ldots k=f(k),$$
$$(k-j-1)\ldots(k-1)=f(k-1),\ \ldots,\ (k-\mu-j)\ldots$$
$$\ldots(k-j)=f(k-j),\ldots.$$

Adding (63) and (64) term by term we have:

$$[\beta_1 k+\ldots+\beta_j k^j+f(k)]C_k^k-$$
$$-[\beta_1(k-1)+\ldots+\beta_j(k-1)^j+f(k-1)]C_k^{k-1}+\ldots+$$
$$+(-1)^\mu[\beta_1(k-\mu)+\ldots+\beta_j(k-\mu)^j+f(k-\mu)]C_k^{k-\mu}+$$
$$+\ldots+(-1)^k[\beta_1\cdot 0+\ldots+\beta_j\cdot 0^j+f(0)]C_k^0=0$$

But by (62) we can write

$$\beta_1 x+\beta_2 x^2+\ldots+\beta_j x^j+f(x)=x^{j+1}.$$

Therefore the above result yields

$$k^{j+1}C_k^k-(k-1)^{j+1}C_k^{k-1}+\ldots+(-1)^\mu(k-\mu)^{j+1}C_k^{k-\mu}+$$
$$+\ldots+(-1)^k\cdot 0^{j+1}\cdot C_k^0=0.$$

This is precisely equality (61) for $m=j+1$. Thus we have demonstrated that Eqs. (61) hold true.

Lastly, consider an arbitrary polynomial whose degree is not higher than $k-1$:

$$P(x)=A_{k-1}x^{k-1}+A_{k-2}x^{k-2}+\ldots+A_0. \qquad (65)$$

Multiplying equalities (61) with $m=0, 1, 2, \ldots, k-1$ by corresponding factors $A_0, A_1, \ldots, A_{k-1}$ we obtain:

$$A_0 C_k^k-A_0 C_k^{k-1}+\ldots+$$
$$+(-1)^\mu A_0 C_k^{k-\mu}+\ldots+(-1)^k A_0 C_k^0=0,$$
$$A_1 k C_k^k-A_1(k-1)C_k^{k-1}+\ldots+$$
$$+(-1)^\mu A_1(k-\mu)C_k^{k-\mu}+\ldots+(-1)^k A_1\cdot 0\cdot C_k^0=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{k-1}k^{k-1}C_k^k-A_{k-1}(k-1)^{k-1}C_k^{k-1}+\ldots+$$
$$+(-1)^\mu A_{k-1}(k-\mu)^{k-1}C_k^{k-\mu}+\ldots+$$
$$+(-1)^k A_{k-1}\cdot 0^{k-1}\cdot C_k^0=0.$$

Adding the equalities term by term we obtain

$$(A_0+A_1 k+\ldots+A_{k-1}k^{k-1})C_k^k-$$
$$-[A_0+A_1(k-1)+\ldots+A_{k-1}(k-1)^{k-1}]C_k^{k-1}+\ldots+$$
$$+(-1)^\mu[A_0+A_1(k-\mu)+\ldots+A_{k-1}(k-\mu)^{k-1}]C_k^{k-\mu}+$$
$$+\ldots+(-1)^k[A_0+A_1\cdot 0+\ldots+A_{k-1}\cdot 0^{k-1}]C_k^0=0,$$

or

$$P(k)\cdot C_k^k-P(k-1)\cdot C_k^{k-1}+\ldots+(-1)^k P(0)\cdot C_k^0=0. \qquad (66)$$

And hence an arbitrary polynomial $P(x)$ whose degree is not higher than $k-1$ satisfies relation (66).

Assume, in particular, $P(x)=(x+n-1)^m$, where n is an arbitrary natural number and m, an integer $0\leqslant m\leqslant k-1$.

Then equality (66) takes the form

$$(k+n-1)^m C_k^k-(k+n-2)^m C_k^{k-1}+\ldots+$$
$$+(-1)^k(n-1)^m C_k^0=0,$$

or multiplying it by α^{k+n-1} and substituting unity for C_k^k:

$$(k+n-1)^m\alpha^{k+n-1}=C_k^{k-1}\alpha(k+n-2)^m\alpha^{k+n-2}-$$
$$-C_k^{k-2}\alpha^2(k+n-3)^m\alpha^{k+n-3}+\ldots+$$
$$+(-1)^{k-1}C_k^0\alpha^k(n-1)^m\alpha^{n-1}. \qquad (67)$$

Comparing (67) with (57) we see that each of the following k sequences satisfies the recursion relation (57):

$$\left.\begin{array}{lr} 1,\ \alpha,\ \alpha^2,\ \ldots,\ \alpha^{n-1},\ \ldots, & (m=0); \\ 0,\ \alpha,\ 2\alpha^2,\ \ldots,\ (n-1)\,\alpha^{n-1},\ \ldots, & (m=1); \\ 0,\ \alpha,\ 2^2\alpha^2,\ \ldots,\ (n-1)^2\,\alpha^{n-1},\ \ldots, & (m=2); \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ 0,\ \alpha,\ 2^{k-1}\alpha^2,\ \ldots,\ (n-1)^{k-1}\,\alpha^{n-1},\ \ldots, & (m=k-1). \end{array}\right\} \quad (68)$$

If we come to establish that these equalities form a basis it will follow that the general term of any sequence which satisfies equation (57) has the form

$$u_n=[B_0+B_1(n-1)+\ldots+B_{k-1}(n-1)^{k-1}]\,\alpha^{n-1}= \\ =Q(n-1)\,\alpha^{n-1}, \quad (69)$$

where $Q(x)=B_0+B_1x+\ldots+B_{k-1}x^{k-1}$ is a polynomial of a degree not higher than $k-1$ with arbitrary coefficients.

In order to prove that (68) is a basis it is sufficient to demonstrate that the following system of k linear equations

$$\begin{gathered} B_0+B_1\cdot 0+\ldots+B_{k-1}\cdot 0^{k-1}=u_1, \\ B_0+B_1\cdot 1+\ldots+B_{k-1}\cdot 1^{k-1}=u_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B_0+B_1(k-1)+\ldots+B_{k-1}(k-1)^{k-1}=u_k \end{gathered}$$

has a solution with respect to the unknowns $B_0, B_1, \ldots, B_{k-1}$ with any $u_1, \ldots, u_k$ (which follows from the proposition stated on p. 19), i.e. that the system

$$\begin{gathered} B_0=0, \\ B_0+B_1+\ldots+B_{k-1}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B_0+(k-1)B_1+\ldots+(k-1)^{k-1}B_{k-1}=0 \end{gathered}$$

has only a zero solution. But the equations of the latter system show that

$$Q(0)=Q(1)=\ldots=Q(k-1)=0,$$

i.e. the equation

$$B_0+B_1x+\ldots+B_{k-1}x^{k-1}=0$$

of a degree not higher than $k-1$ has at least k different

roots: 0, 1, 2, ..., $k-1$. Hence

$$B_0=B_1=\ldots=B_{k-1}=0,$$

and thus we have proved that sequences (68) form the basis of the recursion relations which satisfy equation (57).

If the recursion sequence satisfies the general equation

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\ldots+a_ku_n\ (a_k\neq 0), \quad (70)$$

its characteristic equation

$$q^k=a_1q^{k-1}+\ldots+a_k \quad (71)$$

can have a root α of multiplicity a, a root β of multiplicity b, ..., a root γ of multiplicity c, and a total of $a+b+\ldots+c=k$ roots.

In this most general case it can be demonstrated that the basis is formed by the following k sequences:

$$\begin{array}{l} 1,\ \alpha,\ \alpha^2,\ \ldots,\ \alpha^{n-1},\ \ldots, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0,\ \alpha,\ 2^{a-1}\alpha^2,\ \ldots,\ (n-1)^{a-1}\,\alpha^{n-1},\ \ldots; \\ 1,\ \beta,\ \beta^2,\ \ldots,\ \beta^{n-1},\ \ldots, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0,\ \beta,\ 2^{b-1}\beta^2,\ \ldots,\ (n-1)^{b-1}\beta^{n-1},\ \ldots; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1,\ \gamma,\ \gamma^2,\ \ldots,\ \gamma^{n-1},\ \ldots, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0,\ \gamma,\ 2^{c-1}\gamma^2,\ \ldots,\ (n-1)^{c-1}\gamma^{n-1},\ \ldots \end{array}$$

Therefore

$$u_n=Q(n-1)\alpha^{n-1}+R(n-1)\beta^{n-1}+\ldots+S(n-1)\gamma^{n-1}, \quad (72)$$

where $Q(x)$, $R(x)$, ..., $S(x)$ are some fixed polynomials of a degree not higher than $a-1, b-1, \ldots, c-1$, respectively.

It follows that *the general term u_n of any recursion sequence has the form of a sum of the products of polynomials with respect to* $(n-1)$ (or, which is the same with respect to n) *by general terms of geometric progressions whose common ratios are equal to the roots of the characteristic equation* (71).

If all the roots of the latter equation are simple, the above polynomials are constants and the general term of

the recursion sequence takes the form of a sum of the terms of geometric progressions.

It can be proved that the converse also holds true, namely, that *any sequence* $\{u_n\}$ *whose general term is expressed by formula* (72) *is a recursion sequence* *.

The corresponding equation (71) is constructed using its roots $\alpha, \beta, \ldots, \gamma$ and their multiplicities $a, b, \ldots, c$ (which are the increased by unity degrees of the polynomials Q, $R, \ldots, S$). Hence the recursion relation (Eq. 70) can be found immediately.

Consider as an example the sequence

$$u_n=(n-1)^2\cdot 2^{n-1}+3^{n-1}$$

Comparing it with (72) we see that the roots of the characteristic equation are: $\alpha=2$, $\beta=3$, α being a multiple root of multiplicity $2+1=3$. Therefore the characteristic equation must take the following form:

$$(q-2)^3(q-3)=q^4-9q^3+30q^2-44q+24=0,$$

and the recursion relation is

$$u_{n+4}=9u_{n+3}-30u_{n+2}+44u_{n+1}-24u_n.$$

It is left for the reader to ascertain that the above sequence satisfies the last equation.

10. We shall illustrate now the results of section **9** by several examples. It was shown in section **2** that the terms of an arithmetic progression satisfy the relation

$$u_{n+2}=2u_{n+1}-u_n,$$

the squares of natural numbers, the relation

$$u_{n+3}=3u_{n+2}-3u_{n+1}+u_n,$$

and the cubes of natural numbers, the relation

$$u_{n+4}=4u_{n+3}-6u_{n+2}+4u_{n+1}-u_n.$$

Obviously all these equations are particular cases of the equation

* The proof of the theorem and its converse, which we omit, can be found in Chapter 4 of the book *Division which Leaves a Remainder in Arithmetic and Algebra*, Moscow-Leningrad, 1949; the theory of recursion sequences is expounded in this book in a manner different from that used in this booklet.

$$u_{n+k}=C_k^{k-1}u_{n+k-1}-C_k^{k-2}u_{n+k-2}+\ldots+(-1)^{k-1}C_k^0u_n, \quad (57')$$

discussed in section **9** (in this case $\alpha=1$).

The general term of any sequence which satisfies this equation must have the following form [cf. formula (69)]:

$$u_n=B_0+B_1(n-1)+\ldots+B_{k-1}(n-1)^{k-1}. \quad (69')$$

In order to find the coefficients $B_0, B_1, \ldots B_{k-1}$ it is sufficient to solve the following system of k linear algebraic equations in k unknowns:

$$\left.\begin{aligned} B_0&=u_1,\\ B_0+B_1+\ldots+B_{k-1}&=u_2,\\ &\ldots\ldots\ldots\ldots\\ B_0+B_1(k-1)+\ldots+B_{k-1}(k-1)^{k-1}&=u_k. \end{aligned}\right\} \quad (73)$$

In the case of an arithmetic progression $k=2$ formula (69′) takes the form

$$u_n=B_0+B_1(n-1),$$

and system (73), the form

$$B_0=u_1, \qquad B_0+B_1=u_2.$$

It can be seen that $B_0=u_1$ is the first term of the progression and $B_1=u_2-u_1=d$, its common difference. It follows that

$$u_n=u_1+d(n-1)$$

which is a well known formula.

There is no need in making calculations for the case of sequences of squares or cubes of natural numbers since we know beforehand that the general terms are $u_n=n^2$ or $u_n=n^3$. However, there is some interest in applying the relations (69′) and (73) to the derivation of formulas for the sum of an arithmetic progression, as well as for the sum of squares or cubes of natural numbers.

In section 4 we proved that if the terms of a sequence $\{u_n\}$ satisfy the equation of the form

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\ldots+a_ku_n,$$

then the sums $\{s_n\}$ of the terms of these sequences ($s_1=u_1$, $s_2=u_1+u_2$, $s_3=u_1+u_2+u_3, \ldots$) satisfy the equation

$$s_{n+k+1}=(1+a_1)s_{n+k}+(a_2-a_1)s_{n+k-1}+\ldots+$$
$$+(a_k-a_{k-1})s_{n+1}-a_ks_n.$$

Obviously for equation (57′) we have

$$a_1 = C_k^1,\quad a_2 = -C_k^2,\ \ldots,\ a_k = (-1)^{k-1}C_k^k.$$

Therefore

$$1 + a_1 = 1 + C_k^1 = C_{k+1}^1,$$
$$a_2 - a_1 = -(C_k^2 + C_k^1) + -C_{k+1}^2,$$
$$a_3 - a_2 = C_k^3 + C_k^2 = C_{k+1}^3,$$
$$a_k - a_{k-1} = (-1)^{k-1}(C_k^k + C_k^{k-1}) = (-1)^{k-1}C_{k+1}^k,$$
$$\cdots\cdots\cdots\cdots$$
$$-a_k = (-1)^k C_k^k = (-1)^k C_{k+1}^{k+1}$$

and the equation for $\{s_n\}$ can be written in the following form

$$s_{n+k+1} = C_{k+1}^1 s_{n+k} - C_{k+1}^2 s_{n+k-1} + \ldots + (-1)^k C_{k+1}^{k+1} s_n,$$

or

$$s_{n+k+1} - C_{k+1}^1 s_{n+k} + C_{k+1}^2 s_{n+k-1} - \ldots + (-1)^{k+1} C_{k+1}^{k+1} s_n = 0.$$

Thus, if a sequence $\{u_n\}$ satisfies the recursion relation (57′) of the k-th order, then the sequence of the corresponding sums $\{s_n\}$ satisfies a relation of the same form but of order $k+1$. In particular, for the arithmetic progression $k=2$, for the sequence of squares of natural numbers $k=3$ and for the sequence of their cubes $k=4$ and it follows that for the sequences of the respective sums we must take k in the above equalities (57′), (69′), (73) greater by unity: 3, 4 and 5.

(a) *The Sum of the Terms of an Arithmetic Progression.* According to what has been said s_n is expressed by formula (69′) (substituting s_n for u_n) with $k=3$. It follows that

$$s_n = B_0 + B_1(n-1) + B_2(n-1)^2.$$

The coefficients B_0, B_1, B_2, are determined from system (73) (also substituting s_n for u_n with $k=3$):

$$B_0 = s_1 = u_1,$$
$$B_0 + B_1 + B_2 = s_2 = u_1 + u_2 = 2u_1 + d,$$
$$B_0 + 2\cdot B_1 + 2^2B_2 = s_3 = u_1 + u_2 + u_3 = 3u_1 + 3d.$$

Solving this system of equations we obtain

$$B_0 = u_1,\ B_1 = u_1 + \frac{1}{2}d,\ B_2 = \frac{1}{2}d.$$

Whence it follows that

$$s_n = u_1 + \left(u_1 + \frac{1}{2}d\right)(n-1) + \frac{1}{2}d(n-1)^2 =$$
$$= nu_1 + \frac{1}{2}d(n-1)n = \frac{n[2u_1 + (n-1)d]}{2} =$$
$$= \frac{n[u_1 + u_1 + (n-1)d]}{2} = \frac{n(u_1+u_n)}{2}.$$

(b) *The Sum of Squares of Natural Numbers.* Taking in formulas (69′) and (73) $k=4$ and substituting s_n for u_n we have

$$s_n = B_0 + B_1(n-1) + B_2(n-1)^2 + B_3(n-1)^3$$

and

$$B_0 = s_1 = 1,$$
$$B_0 + B_1 + B_2 + B_3 = s_2 = 1 + 2^2 = 5,$$
$$B_0 + 2B_1 + 4B_2 + 8B_3 = s_3 = 1 + 2^2 + 3^2 = 14,$$
$$B_0 + 3B_1 + 9B_2 + 27B_3 = s_4 = 1 + 2^2 + 3^2 + 4^2 = 30.$$

The last system yields

$$B_0 = 1,\ B_1 = 2\frac{1}{6},\ B_2 = 1\frac{1}{2},\ B_3 = \frac{1}{3}.$$

Therefore

$$s_n = 1 + \frac{13}{6}(n-1) + \frac{3}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 =$$
$$= \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{n(1+3n+2n^2)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

We have obtained a well known formula.

(c) *The Sum of Cubes of Natural Numbers.* This sum is expressed by the formula

$$s_n = \frac{n^2(n+1)^2}{4}.$$

We leave the derivation of this formula for the reader as an exercise.

In conclusion let us consider one more example, the sequence α, $2\alpha^2$, $3\alpha^3$, ..., $n\alpha^n$, ... ($\alpha \neq 0$, $\alpha \neq 1$).

In this sequence

$$u_n = n\alpha^n \quad (n = 1, 2, 3, \ldots).$$

It is easily seen that

$$u_{n+2} = 2\alpha u_{n+1} - \alpha^2 u_n.$$

Indeed

$$2\alpha u_{n+1}-\alpha^2 u_n=2\alpha(n+1)\alpha^{n+1}-\alpha^2 n\alpha^n=$$
$$=(n+2)\alpha^{n+2}=u_{n+2}.$$

Since $k=2$, $a_1=2\alpha$ and $a_2=-\alpha^2$, the sequence consisting of sums $\{s_n\}$ ($s_1=\alpha$, $s_2=\alpha+2\alpha^2$, $s_3=\alpha+2\alpha^2+3\alpha^3$, ...) satisfies the equation [cf. (30)]:

$$s_{n+3}=(a_1+1)s_{n+2}+(a_2-a_1)s_{n+1}-a_2s_n=$$
$$=(2\alpha+1)s_{n+2}-(\alpha^2+2\alpha)s_{n+1}+\alpha^2s_n.$$

The corresponding characteristic equation is as follows:

$$q^3=(2\alpha+1)q^2-(\alpha^2+2\alpha)q+\alpha^2.$$

It can be easily ascertained that this equation is satisfied with $q=\alpha$. Dividing the polynomial $q^3-(2\alpha+1)q^2+(\alpha^2+2\alpha)q-\alpha^2$ by $q-\alpha$ we obtain the quotient

$$q^2-(\alpha+1)q+\alpha.$$

For this reason the two remaining roots of the characteristic equation satisfy the following equation:

$$q^2-(\alpha+1)q+\alpha=0.$$

These roots are α and 1.

Thus the characteristic equation has a multiple root α of multiplicity $a=2$ and a simple root $\beta=1$.

Therefore we obtain for s_n [cf. formula (69) in which we must write s_n instead of u_n, $\alpha=\alpha$, $Q(x)=B_0+B_1x$, i.e. a polynomial of degree 1, $\beta=1$ and $R(x)=C_0$, a constant]:

$$s_n=[B_0+B_1(n-1)]\alpha^{n-1}+C_0 \quad (n=1, 2, 3, \ldots).$$

The coefficients B_0, B_1 and C_0 are found from the following system of equations with $n=1$, 2 and 3, respectively:

$$B_0+C_0=s_1=\alpha, \quad (B_0+B_1)\alpha+C_0=s_2=\alpha+2\alpha^2,$$
$$(B_0+2B_1)\alpha^2+C_0=s_3=\alpha+2\alpha^2+3\alpha^3.$$

Hence

$$B_0=\frac{\alpha^3-2\alpha^2}{(\alpha-1)^2}, \quad B_1=\frac{\alpha^2}{\alpha-1} \quad \text{and} \quad C_0=\frac{\alpha}{(\alpha-1)^2}.$$

Consequently

$$s_n=[B_0+B_1(n-1)]\alpha^{n-1}+C_0=\frac{n\alpha^{n+2}-(n+1)\alpha^{n+1}+\alpha}{(\alpha-1)^2}=$$
$$=\frac{u_n\alpha^2-(u_{n+1}-u_n)}{(\alpha-1)^2}.$$

CONCLUSION

This book is meant to give the reader an idea of the variety of recursion sequences and of their place in mathematics. Along with it it was shown that the recursion sequences do not greatly differ from their simplest forms — the geometric progression and the sequences whose terms are powers of natural numbers (in particular, the sequences of natural numbers as such being an arithmetic progression); the recursion sequences can be expressed making use of these simplest sequences.

However even in elementary mathematics sequences are very often met with which are not recursion sequences. One of these is the sequence of prime numbers, one of the most important in mathematics:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

This sequence with its deep and complex properties is a subject of the theory of numbers.

Sequences of values of many elementary functions are not recursion sequences as well. For instance, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots;$$

(this is the sequence of values of the function $y=\frac{1}{x}$ with $x=1, 2, 3, \ldots$), or the sequences

$$1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots,$$
$$\log 1, \log 2, \log 3, \log 4, \ldots, \log n, \ldots$$

(these are the sequences of values of the functions $\sqrt{x}$ and $\log x$) etc.

These and similar sequences* (including the recursion sequences) are the subject of the branch of mathematics mentioned above — the calculus of finite differences.

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* Sequences of values of the so-called analytical functions are meant whose simplest instances are the elementary functions.

And finally, in elementary mathematics and particularly in analysis studied in colleges and universities a very important place is occupied by convergent sequences, i. e. sequences which have finite limits. Their study is the most important problem in the theory of limits and belongs to the foundations of analysis. The properties of individual terms of the sequences are even less than secondary: only the existence of a limit and its value are of consequence.

We consider these remarks to be necessary in order to bring home to the reader that the theory of recursion sequences given in this book from the viewpoint of its subject-matter as well as from the viewpoint of the relationships that result from it is only a particular and very modest chapter in the theory of sequences.

TO THE READER

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Our address:
USSR, 129820, Moscow 1-110, GSP
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